Institution: University of Kent

Unit of Assessment: 10 (Mathematical Sciences)

Title of case study: The Digital Library of Mathematical Functions as a public resource

1. Summary of the impact

The US National Institute of Science & Technology (NIST) "Digital Library of Mathematical Functions" (DLMF) available at http://dlmf.nist.gov/ is an online resource which informs the general public by making detailed properties of elementary and higher functions freely available to all. The DLMF together with its print companion, the new NIST Handbook of Mathematical Functions, published by Cambridge University Press (CUP, 2010), replaces and updates the National Bureau of Standards Handbook of Mathematical Functions (Abramowitz & Stegun), a classic source which is one of the best-selling mathematics texts of all time. The fact that the 10-year DLMF project has been led and funded by NIST, part of the US Department of Commerce, is a measure of the economic importance of making this information publically available, so that science and engineering practitioners can use it freely. In addition, CUP has already received considerable benefit through the strong sales of the NIST Handbook since its publication, and prestige from the praise that the DLMF has already received from the US government and industrial societies.

A chapter of the new Handbook has been contributed by a member of the School of Mathematics, Statistics and Actuarial Science, and is based on research carried out at Kent. The provision of this information in an interactive format raises awareness and understanding, and enhances the work of practitioners such as industrial scientists and teachers in disciplines outside mathematics, by allowing them easy and immediate access to the most relevant and up to date research results in this area.

2. Underpinning research

The research that contributed to the DLMF project and the NIST Handbook was carried out at Kent by **Clarkson** (1995-), as well as by **Mansfield** (1995-) and PhD student **Webster** (1995-1998).

The original National Bureau of Standards Handbook (Abramowitz & Stegun) was first published in 1964, and was subsequently reprinted many times. One of the main reasons for producing the DLMF and the completely new NIST Handbook was that research into mathematical functions has advanced significantly, meaning not only that new information on classical special functions needed to be presented, but also that new special functions needed to be included, which were not even mentioned in the old Handbook.

Chapter 32, Painlevé Transcendents, contributed by **Clarkson** (sole author), concerns a topic that was absent from Abramowitz & Stegun's Handbook, whose relevance to the modelling of scientific problems involving nonlinearity has only begun to be appreciated in the last few decades.

Almost all classical special functions are solutions of linear differential equations, and indeed linear equations provide accurate models in a huge variety of contexts; particular examples in the NIST Handbook include the intensity profile of light in a rainbow (Airy functions), and Kelvin's wave pattern in the wake of a ship. However, in the latter half of the twentieth century, our understanding of nonlinear models has increased substantially, and it is now well appreciated that there are essential features of nonlinear systems that cannot be captured by linear approximations.

Painlevé transcendents are solutions of special nonlinear ordinary differential equations, and they appear in models for many different natural and man-made phenomena, including energy levels and scattering off heavy nuclei, self-similar behaviour in wave-breaking, and the statistics of the bus delivery system in Cuernavaca, Mexico (among many other examples). **Clarkson** has worked on the detailed properties of Painlevé equations for several decades, including from 1995 at the University of Kent, and is acknowledged as a world expert on the subject, which is why he was chosen by NIST as the author of the chapter for the DLMF and the NIST Handbook.

Of particular importance are explicit solutions of Painlevé equations, which occur for special parameter values. In research carried out at Kent, **Mansfield** and **Webster [3.2]** constructed one-parameter families of solutions of the third Painlevé equation, while subsequent work by **Clarkson** and **Mansfield** concerned the second Painlevé equation and the specific structure of its rational

solutions **[3.3],** which have applications in nonlinear wave equations admitting solitons **[3.4]**. Earlier work by Bassom, **Clarkson**, Law and McLeod concerned uniform asymptotics for the second Painlevé transcendent **[3.1]**. These works were specifically cited in Chapter 32 of the NIST Handbook.

The general properties of Painlevé equations (Hamiltonian structure, asymptotics, special solutions) allow them to be viewed as nonlinear special functions. The substantial survey article **[3.5]** highlights these properties and pinpoints their relevance in a wide variety of application areas. This article underpins the contribution by Clarkson to the DLMF and the NIST Handbook.

3. References to the research

[3.1]* A P Bassom, **P A Clarkson**, C K Law and J B McLeod, *Application of uniform asymptotics to the second Painlevé transcendent*, Archive for Rational Mechanics and Analysis, 143 (1998) 241-271. doi: 10.1007/s002050050105

[3.2] **E L Mansfield** and **H N Webster**, *On one-parameter families of Painlevé III*, Studies in Applied Mathematics, 101 (1998) 321-341. doi: 10.1111/1467-9590.00096

[3.3]* P A Clarkson and **E L Mansfield**, *The second Painlevé equation, its hierarchy and associated special polynomials*, Nonlinearity, 16 (2003) R1-R26. doi: 10.1088/0951-7715/16/3/201

[3.4]* P A Clarkson, *Special polynomials associated with rational solutions of the Painlevé equations and applications to soliton equations*, Computational Methods and Function Theory, 6 (2006) 329-401. doi: 10.1007/BF03321618

[3.5] **P A Clarkson**, *Painlevé equations - nonlinear special functions*, in "Orthogonal Polynomials and Special Functions: Computation and Application", Editors F Marcellán and W van Assche, Lecture Notes in Mathematics, 1883, Springer-Verlag, Berlin (2006) pp. 331-411. doi: 10.1007/978-3-540-36716-1_7

(References marked with a star best indicate the quality of the underpinning research.)

4. Details of the impact

The NIST Digital Library of Mathematical Functions makes the properties of transcendental functions freely available to the general public. The online, interactive format of the Digital Library is a new feature, with its print companion, the new Handbook, providing further detailed information for practitioners such as industrial scientists. This has a commercial impact, in particular on the US economy and on the publisher of the Handbook (Cambridge University Press). It also informs the practice of teachers and researchers outside mathematics, and engineers and professionals outside academia.

Professionals in all fields of science and engineering need reliable and detailed technical information to be readily available, in order to develop specific industrial applications and carry out research. In the past, such information was collected into handbooks, which would be revised and reprinted according to demand. Arguably the most successful mathematical handbook of all time is the National Bureau of Standards Handbook of Mathematical Functions (Abramowitz & Stegun), first published in 1964 **[5.7].** Although still a classic, it has gradually become outdated, as important new properties of special functions, and new families of such functions, have been discovered over the last fifty years. Nevertheless, it has served as a template for a new project, the Digital Library of Mathematical Functions, led by NIST.

The aim of the NIST Digital Library is to make carefully selected and accurate data on the elementary and higher mathematical functions available to the public, with experienced professionals being the target audience. The primary criterion for inclusion of information is usefulness in disciplines outside mathematics. As well as the online hypertext version, which includes interactive graphics and search tools, there is a more traditional print version, the NIST Handbook of Mathematical Functions, published by Cambridge University Press (2010). According to the General Editor, Daniel Lozier at NIST, "The Handbook will be used by anyone who is doing a project in mathematics, science or engineering" **[5.7].**

The fact that the Digital Library of Mathematical Functions is funded by NIST, a US government

agency, which is part of the US Department of Commerce, clearly indicates the importance of this information to scientific professionals. As well as their immediate uses in mathematics and physics (for example, in describing the pattern of waves in the wake of a moving ship **[5.7]**), mathematical functions are applied in manufacturing technology, civil engineering, bioscience, electronics, and computer networks. The impact of the NIST Digital Library on these areas has so far been measured indirectly, in two different ways. Since the site http://dlmf.nist.gov/ was opened, NIST estimates that **more than half of the downloads were by non-academic users [5.1].** The web usage data analysis of NIST **[5.2]** shows in particular that Chapter 32 written by **Clarkson** was downloaded more than 2000 times per year, so that NIST estimates that more than 1000 non-academic users are downloading **Clarkson**'s chapter each year. At the request of NIST, a citation analysis was performed to compare usage of the old NBS Handbook (Abramowitz & Stegun) and the new NIST Handbook **[5.1]**. The latter analysis shows that while the NBS Handbook is still very widely used, the amount of usage has gradually started to decline since the appearance of the NIST Handbook: citations to the old Handbook increased by 5% from 2007-2009, but decreased by 12% from 2010-2012; while citations to the new Handbook more than doubled from 2011-2012. According to NIST, 15 of the top 20 journals that cite the new NIST Handbook of Mathematical Functions are **not** Mathematical journals, confirming the role of the NIST Handbook of Mathematical Functions as a **tool for scientists and engineers**.

The new Handbook has already gained more than 600 citations (according to Google Scholar on 31st July 2013 **[5.8]**), including in the following disciplines: Chemistry, Forest Research, Optics, Water Resources. According to Cambridge University Press (CUP), the NIST Handbook is expected to become a bestseller just like the old Handbook, and CUP receives **considerable economic benefit and prestige** by publishing the Handbook. This is confirmed by the Editorial Director of CUP: "Since publication in 2010, the Handbook has seen **strong sales** throughout the world, especially in North America and Europe, has attracted much praise for quality and breadth of content, and for its high production values. It is not just the print version of the book that users appreciate; it is also available as a browsable CD that comes packaged with the book and as an addressable database online. These features have extended the usability of the Handbook over its predecessor and made it a resource that can be readily **used worldwide**, not just by mathematicians, but in fundamental science, applied science and engineering." **[5.10]** This is backed up by several reviews in industrial and engineering publications. For instance, the US Society for Industrial and Applied Mathematics (SIAM) wrote that "The NIST Handbook is indeed a monumental achievement, and the many, many individuals who participated in its creation and dissemination are to be congratulated and thanked" **[5.11].** And Optics and Photonics News recognized that "The National Institute of Standards and Technology (NIST) and Cambridge University Press are to be congratulated for publishing a treasury. It is eminently readable with clear, sharp, high-contrast text, mathematical notation and colored graphs and figures. People who work with functions will delight in this handbook" **[5.11].**

According to statements from a report to the US House of Representatives, "NIST carries out in a superb fashion an absolutely vital role in supporting as well as facilitating the further development of the technological base of the U.S. economy" and, in particular, "The Digital Library of Mathematical Functions is without peer in the broader community" **[5.9]**. This acknowledges the significant economic impact of the DLMF.

The innovative nature of the Digital Library, in the way that it makes the properties of transcendental functions available online, has already been acknowledged in US Government circles. The DLMF was one of only 10 projects (out of 200 nominated) to win a 2011 Government Computer News (GCN) Award for Outstanding Information Technology Achievement in Government **[5.3]**. This acknowledges the high quality and easy accessibility of the information provided and further recognition by US Government.

The Digital Library is now used not only to enhance academic knowledge but also as a resource which informs the teaching practice of many disciplines outside mathematics, hence changing practice in the HE sector internationally. For instance, in the Department of Astronomy of the University of Tokyo the Digital Library is used to teach Celestial Mechanics **[5.4]**; and in the School

of Engineering of the Catholic University of America it is used to teach Acoustic Metrology **[5.5]**. (These examples are indicative of the immediate impact of the Handbook on teaching practice in higher education; other examples include teaching at Princeton [Astronomy, Observing and Modeling the Universe], and at Colorado State University [Mechanical Engineering Problem Solving].) The Digital Library is also used as a standard reference by the US army **[5.6]**.

Summary: The research on Painlevé equations at Kent has contributed to the wealth of the US and other economies, and CUP, via the DLMF and the new Handbook. The DLMF is used widely by scientists and practitioners, and has attracted much praise contributing to the prestige of CUP. It has also influenced teaching practice in disciplines other than mathematics across the globe.

5. Sources to corroborate the impact

[5.1] Citation Analysis for the NIST Handbook of Mathematical Functions, 2007-2012, from 24th July 2013, showing wide usage outside academia during the REF period. (See Contact 2.)

[5.2] Downloads data analysis for Chapter 32 of the Digital Library of Mathematical Functions, 2011-2012, from 29th May 2013, showing wide usage outside academia during the REF period. (See Contact 2.)

[5.3] Award from the US Government Computer News magazine: http://www.nist.gov/itl/math/dlmf-102511.cfm

[5.4] Reading list for Celestial Mechanics, Department of Astronomy, University of Tokyo: http://www.s.u-tokyo.ac.jp/en/current/syllabus/pdf/2012/astron.pdf

[5.5] Syllabus in Acoustic Metrology, School of Engineering, Catholic University of America: http://faculty.cua.edu/vignola/Vignola_CUA/ME_661_files/ME%20661%20syllabus%20Summer%202011.pdf

[5.6] US army report on standards for measurements and mathematical constants confirming that the Digital Library of Mathematical Functions is a standard reference for the US army. This report is available at: http://www.wsmr.army.mil/RCCsite/Documents/156-10_%20Physical%20Constants,%20Units,%20and%20Uncertainty%20Standard/156-10_%20Physical%20Constants,%20Units,%20and%20Uncertainty%20Standard.pdf

[5.7] The Birth of a Classic…Take Two: NIST video available at http://www.youtube.com/watch?v=Exf02R1FnXY showing that the original handbook is the most cited NIST resource and the updated 2010 online version is more usable for science professionals.

[5.8] Google scholar entry for the Handbook of Mathematical Functions showing the wide usage of the Handbook outside Mathematics: http://scholar.google.com/scholar?cluster=10873204128209726657&hl=en&as_sdt=2005&sciodt=0,5

[5.9] Statement of Professor Ross B. Corotis before the Subcommittee on Technology Committee on Science, Space, and Technology, US House of Representatives, March 20, 2013, confirming the unique role of the DLMF in the broader community. http://science.house.gov/sites/republicans.science.house.gov/files/documents/HHRG-113-SY19-WState-RCorotis-20130320.pdf

[5.10] Email from an Editorial Director of CUP confirming the benefits and prestige gained from CUP by publishing the NIST Handbook of Mathematical Functions. (See Contact 3.)

[5.11] Reviews on the NIST Handbook of Mathematical Functions and DLMF confirming the prestige gained by CUP in publishing this book are available at: http://www.cambridge.org/pa/knowledge/isbn/printView/item6005277/?site_locale=es_PA

If any of the links above is no longer working, pdf copies are available on request.